

Exercise Sheet 10

Exercise 1. Consider the congruence subgroup

$$\Gamma(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{3} \right\}$$

and its action on \mathbb{H} by Möbius transformations. Show that the quotient $\mathbb{H}/\Gamma(3)$ is a Riemann surface and that $\pi_1(\mathbb{H}/\Gamma(3)) \cong \Gamma(3)$.

Exercise 2. Let $f : X \rightarrow Y$ be a covering map between connected spaces and suppose that Y is a Riemann surface. Show that X can be endowed with a (unique) Riemann surface structure such that f is holomorphic and that all covering transformations of f are biholomorphisms.

Exercise 3. Show that every connected n -sheeted covering of the punctured disc \mathbb{D}^* is isomorphic as a covering to

$$p_n : \mathbb{D}^* \rightarrow \mathbb{D}^*, \quad z \mapsto z^n.$$

Exercise 4. (for credit, due on 30 November) (5 points)

Let G be a finite group. Show that there exists a compact Riemann surface X , a finite set of points $B \subset \mathbb{P}^1$, and a holomorphic map $f : X \rightarrow \mathbb{P}^1$ with branch locus B such that the deck group of the restriction

$$f|_{X \setminus f^{-1}(B)} : X \setminus f^{-1}(B) \rightarrow \mathbb{P}^1 \setminus B$$

is isomorphic to G . **Hint:** Let G be generated by g_1, \dots, g_r . Realize G as a quotient of $\pi_1(\mathbb{P}^1 \setminus \{r+1 \text{ points}\})$.

Exercise 5. Let $f : X \rightarrow Y$ be a covering map between compact Riemann surfaces. Show that f is a Galois covering if and only if the field extension $\mathcal{M}(X)/\mathcal{M}(Y)$ is Galois. In this case $\text{Gal}(\mathcal{M}(X)/\mathcal{M}(Y)) \cong \text{Deck}(f)$.